On Hawking radiation of $3D$ Rotating Hairy Black Holes

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Abstract

We study the Hawking radiation of $3D$ rotating hairy black holes. More concretely, we compute the transition probability of a bosonic and fermionic particle in such backgrounds. Thew, we show that the transition probability is independent of the nature of the particle. It is observed that the charge of the scalar hair B and the rotation parameter a control such a probability.

The study of $(1+2)D$-dimensional black holes provides a good understanding of low-dimensional gravity models and their related quantum field theories. The significance comes from the evaporation of the black hole considered as a consequence of Hawking radiation [1, 2, 3, 4, 5, 6, 7, 8, 9].

The $(1+2)D$ hairy black holes and their thermodynamical properties are extensively investigated using different methods [10, 11, 12, 13, 14]. More precisely, their statistical physics [15] and critical behavior have been studied by considering the cosmological constant as a thermodynamical pressure [16, 17, 18, 19, 20, 21]. It is interesting to recall the connection between such $3D$ black holes and non-commutative geometry which can be found in [22, 23] and with the CFT correspondence as proposed in [24, 25].
The aim of this work is to contribute to this program by exploring the corresponding Hawking radiation. After an introduction of the rotating hairy $(2+1)D$ black hole, we discuss the tunneling of the bosonic and fermionic particles from such black holes. Then, we show that the probability of the transition is independent of the the nature of the particle.

To start, we reconsider the study of $3D$-dimensional gravity with a non-minimally coupled scalar field. The black hole solution is dubbed hairy black hole in three dimensions. In the absence of the Maxwell gauge fields, the corresponding model is controlled by the following action [12]

$$\mathcal{I}_{\mathcal{R}} = \frac{1}{2}\int d^3x\sqrt{-g}\left[R - g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \xi R\phi^2 - 2V(\phi)\right], \tag{1}$$

where ϕ is the dynamical scalar field. For simplicity reason, we elaborate a particular situation where the coupling and the gravitational constants are fixed to $\xi = \frac{1}{8}$ and $\kappa = 8\pi G = 1$ respectively. Within this assumption, the metric solution takes the following form

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2\left(d\theta + \omega(r)dt\right)^2. \tag{2}$$

The functions f and w, describing such a black hole solution, read as

$$f(r) = 3\beta + \frac{2B\beta}{r} + \frac{(3r+2B)^2a^2}{r^4} + \frac{r^2}{\ell^2} \tag{3}$$

$$\omega(r) = -\frac{(3r+2B)a}{r^3} \tag{4}$$

where β is identified with $-\frac{M}{3}$. In these equations, a denotes the rotating angular momentum parameter. B is linked with the dynamical scalar field via the relation

$$\phi(r) = \pm\sqrt{\frac{8B}{r+B}}. \tag{5}$$

For later use, an explicit form of the potential are needed. In the present study, we use

the potential $V(\phi)$ proposed in [13] and given by

$$V(\phi) = \frac{1}{512}\left(\frac{a^2\left(\phi^6 - 40\phi^4 + 640\phi^2 - 4608\right)\phi^{10}}{B^4\left(\phi^2 - 8\right)^5} + \phi^6\left(\frac{\beta}{B^2} + \frac{1}{\ell^2}\right) + \frac{1024}{\ell^2}\right). \tag{6}$$

Concretely, we discuss the Hawking radiation of the corresponding black solution. Indeed, we compute the probability transition of bosonic and fermerionic particles in such black hole backgrounds.

Here, we deal with the emission of scalar particles from rotating hairy three dimensional black hole as tunneling phenomena across the event horizon. This can be done by solving the Klein-Gordon equation for the the scalar Ψ. This equation reads as which reads

$$g^{\mu\nu}\partial_\mu\partial_\nu\Psi - \frac{m^2}{\hbar^2}\Psi = 0 \tag{7}$$

where μ and ν take the values $0, 1, 2$ for the coordinates t,r,θ respectively. The quantity m represents the mass of the particle. In what follows, we use WKB approximation and choose an ansatz of the form

$$\Psi(t, r, \phi) = e^{\frac{i}{\hbar}I(t,r,\theta)+I_1(t,r,\theta)+\mathcal{O}(\hbar)} \tag{8}$$

Using (7), the exponential term and multiplying by $\hbar^2$, we obtain

$$g^{tt}(\partial_t I)^2 + g^{rr}(\partial_r I)^2 + g^{t\theta}(\partial_t I\partial_\theta I) + g^{\theta\theta}(\partial_\theta I)^2 + m^2 = 0. \tag{9}$$

where $g^{\mu\nu}$ are the components of the metric (2). Exploring the symmetries of such black holes, ∂_t and ∂_ϕ are considered as the Killing fields. Thus, there exists a solution for this differential equation given in terms of the classical action I. The later can be written as follows

$$I = \zeta t + W(r) + j\theta + K, \tag{10}$$

where w and j represent the energy and angulaire momentum of the particle respectively. K is a constant which could be complex. Using the above expressions, we arrive to

$$W'(r) = \pm\sqrt{-\frac{g_{tt}}{g_{rr}}\left(\zeta^2 + \frac{g^{\theta\theta}}{g^{tt}}j^2 - \frac{g^{t\theta}}{g_{tt}}j\zeta + \frac{1}{g^{tt}}m^2\right)} \tag{11}$$

Substituting the value of the metric tensor, we obtain the following integral

$$W_\pm(r) = \pm\int\frac{\sqrt{\zeta^2 - (\frac{f(r)}{r^2} - \omega^2)j^2 + 2\omega\zeta j - f(r)m^2}}{f(r)}dr. \tag{12}$$

We observe a simple pole at $r = r_+$. From the residue theory, dealing with semi circles,

the performed integration results in,

$$W_{\pm} = \pm \pi i \frac{\sqrt{\zeta^2 + (\omega(r_+))^2 j^2 + 2\omega(r_+)\zeta j}}{f'(r)}. \tag{13}$$

Therefore, this equation implies that

$$\begin{aligned} ImW_+ &= \pi \frac{\zeta + j\omega(r_+))}{f'(r_+)} & (14) \\ &= \frac{3\pi r_+^2 \ell^2 \left(\zeta r_+^3 - 2aj\left(B + r_+\right)\right)}{\ell^2 \left(2BMr_+^3 - 3a^2\left(8B + 9r_+\right)\right) + 6r_+^6}. & (15) \end{aligned}$$

It is recalled that the Hawking radiation can be viewed as a process of quantum tunneling of particles from the black hole horizon. From this point of view, we compute the imaginary part of the classical action for this classically forbidden process of emission across the horizon. In this semi-classical approach, the probabilities of the crossing the horizon from inside to outside and from outside to inside, reads as [26, 27]

$$\begin{aligned} P_{emission} &\propto \exp\left(\frac{-2}{\hbar} ImI\right) = \exp\left(\frac{-2}{\hbar}(ImW_+ + ImK)\right). & (16) \\ P_{absorption} &\propto \exp\left(\frac{-2}{\hbar} ImI\right) = \exp\left(\frac{-2}{\hbar}(ImW_- + ImK)\right). & (17) \end{aligned}$$

It is known that any outside particle will certainly fall into the black hole. Thus, we should take $ImK = -ImW_-$. Using (13), we also have $W_+ = -W_-$. This indicates that the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma = \exp\left(\frac{-4}{\hbar} ImW_+\right). \tag{18}$$

A priori there are many ways, including the first thermodynamical law, to get the tunneling probability of scalar field. However we remark, from (14), that ImW_+ is related to the temperature hidden in the $f'(r_+)$ expression. Using Boltzmann factor and substituting (14) into (18), we get the desired formula,

$$\Gamma = \exp\left[-\frac{12\pi r_+^2 \ell^2 \left(\zeta r_+^3 - 2aj\left(B + r_+\right)\right)}{\hbar\left(\ell^2 \left(2BMr_+^3 - 3a^2\left(8B + 9r_+\right)\right) + 6r_+^6\right)}\right]. \tag{19}$$

This quantity represents the tunneling probability of scalar particle from inside to outside the event horizon of rotating hairy black hole. Comparing this equation with $\Gamma = \exp(-\beta\omega)$, which is Boltzmann factor for particle of energy ω where β is the inverse of the temperature of the horizon [26, 27], it is possible to derive the Hawking temperature. Notice that one can also compute the Hawking temperature directly from the equation,

$$T_H = \frac{f'(r_+)}{4\pi}|_{r=r_+}. \tag{20}$$

and it is easy to cheek that

$$T_H = \frac{3\left(4a^2B^2\ell^2 + 12a^2Br\ell^2 + 9a^2r^2\ell^2 + r^6\right)}{r^3\ell^2(2B+3r)}. \tag{21}$$

Besides, similar form to (21) can be obtained from the entropy using the first law of the thermodynamics.

It is interesting to note here that we can also recover the formula given in [10]. Furthermore, for $B = 0$ the result of [28] is easily reproduced.

Having discussed the bosonic equations, we move now discuss the fermionic particles. In this case, we compute Hawking radiation from rotating three dimensional hairy black hole. Precisely, we consider the two component massive spinor field Ψ, with mass m, verifying the following Dirac equation

$$i\hbar\gamma^\alpha e_a^\mu \nabla_\mu \Psi - m\Psi = 0. \tag{22}$$

∇ is the spinor covariance derivative given by $\nabla_\mu = \partial_\mu + \Omega_\mu$, with

$$\Omega_\mu = \frac{i}{2}\Gamma_\mu^{\alpha\beta}\Sigma_{\alpha\beta}, \tag{23}$$

$$\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta], \qquad \Omega_\mu = \frac{-1}{8}\Gamma_\mu^{\alpha\beta}[\gamma^\alpha, \gamma^\beta]. \tag{24}$$

The γ matrices, in three space-time dimensions, take the following form

$$\gamma^a = (-i\sigma^1, \sigma^0, \sigma^2) \tag{25}$$

where σ^i are the Pauli sigma matrices, and where e_a^μ are the vielbein fields. Choosing the curved space γ^μ matrices as

$$\gamma^t = \begin{pmatrix} 0 & -\frac{1}{\sqrt{f}} \\ \frac{1}{\sqrt{f}} & 0 \end{pmatrix}, \quad \gamma^r = \begin{pmatrix} 0 & \sqrt{f} \\ \sqrt{f} & 0 \end{pmatrix}, \quad \gamma^\theta = \begin{pmatrix} \frac{1}{r} & \frac{8\omega}{\sqrt{f}} \\ -\frac{8\omega}{\sqrt{f}} & \frac{1}{r} \end{pmatrix} \tag{26}$$

satisfying the condition $\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}\mathbb{1}$, where $\mathbb{1}$ is the identity matrix, the equation of motion reads as

$$i\left(\gamma^t\partial_t + \gamma^r\partial_r + \gamma^\theta\partial_\theta\right)\Psi - \frac{m}{\hbar}\Psi = 0. \tag{27}$$

For a fermionic particle with spin $1/2$, we have two states namely spin-up ($\uparrow$) and spin-down ($\downarrow$). Thus, we can use the following ansatz for the solution

$$\Psi_\uparrow = \begin{pmatrix} X(t,r,\theta) \\ 0 \end{pmatrix} e^{\frac{i}{\hbar}I_\uparrow(t,r,\theta)} \tag{28}$$

$$\Psi_\downarrow = \begin{pmatrix} 0 \\ Y(t,r,\theta) \end{pmatrix} e^{\frac{i}{\hbar}I_\downarrow(t,r,\theta)} \tag{29}$$

where $\Psi_\uparrow$ denotes the wave function of the spin-up particle and $\Psi_\downarrow$ is for the spin-down case. Inserting equation (28) for the spin-up particle into the Dirac equation (27), then dividing by exponential term and multiplying by $\hbar$, we get the following equation

$$-\frac{X}{\sqrt{f}}\partial_\uparrow I_\uparrow(t,r,\theta)-\sqrt{f}\,\partial_r I_\uparrow(t,r,\theta)+\frac{\omega}{\sqrt{f}}\partial_\theta I_\uparrow(t,r,\theta)=0. \tag{30}$$

Notice here that the separation of variables for the spin-up case can be used to produce Killing vectors $\left(\frac{\partial}{\partial t}\right)^\mu$ and $\left(\frac{\partial}{\partial\theta}\right)^\mu$. To handle the above equation, we should express $I_\uparrow(t,r,\theta)$ as

$$I_\uparrow(t,r,\theta)=-\zeta t+W(r)+\Theta(\theta)+K=-\zeta t+W(r)+j\theta+K \tag{31}$$

where ζ and j are the energy and the angular momentum of the emitted particle respectively. K is a complex constant. Using the expression in the above equation, we obtain

$$\frac{X}{\sqrt{f}}\zeta-\sqrt{f}\,X\,\partial_r W-\frac{\omega}{\sqrt{f}}\partial_\theta\Theta=0. \tag{32}$$

Putting the equation (31) in (32), we get

$$\frac{\zeta}{\sqrt{f}}-\sqrt{f}\,X\,\partial_r W-j\frac{\omega}{\sqrt{f}}=0 \tag{33}$$

where

$$\partial_r W=\frac{1}{f}\left(\zeta-j\omega\right) \tag{34}$$

For spin-down particle, the phase $I_\downarrow$ and its r-dependence can be obtained using similar method and steps. More precisely, we get expressions similar to those shown in equations (31) and (34). Using (34), we obtain

$$W=\int\frac{dr}{f}(\zeta-j\,\omega). \tag{35}$$

Then, we integrate along a semi circle around the pole at $r_+=0$. At the horizon of the black hole, the radial function is then given by

$$W=\frac{3\pi r_+^2\ell^2\left(\zeta r_+^3-2aj\left(B+r_+\right)\right)}{\ell^2\left(2BMr_+^3-3a^2\left(8B+9r_+\right)\right)+6r_+^6}. \tag{36}$$

Similar calculation shows that tunneling probability takes the form (18)

$$\Gamma=\exp\left[-\frac{12\pi r_+^2\ell^2\left(\zeta r_+^3-2aj\left(B+r_+\right)\right)}{\hbar\left(\ell^2\left(2BMr_+^3-3a^2\left(8B+9r_+\right)\right)+6r_+^6\right)}\right]. \tag{37}$$

Remarkably we find the same expression as the one obtained by solving the Klein-Gordon equation. It is recalled that similar steps to those in the bosonic case have also been used.

Again note that if we set $B = 0$, we recover the result found in [28].

In this work, we have studied the Hawking radiation of $3D$ rotating hairy black holes. In particular, we have computed the transition probability of bosonic and fermionic particles in such backgrounds. Then, we have shown that the transition probability is independent of the nature of the particle. It should be interesting to extend this work to higher dimensional cases.

References

References

[1] S. W. Hawking, *Particle creation by black holes*, Commun.Math.Phys.**43** (1975)199.

[2] G. W.Gibbons, S. W. Hawking,*Cosmological event horizons, thermodynamics, and particle creation*, Phys.Rev. **D15** (1977)2738.

[3] H. L. Li, S. Z. Yang, *Hawking radiation from the charged BTZ black hole with backreaction*, EPL.**79**(2007)20001.

[4] L. Yan, H. Wei, *Solution of Dirac equation around a charged rotating black hole*, Chinese Physics **B**, 2014, 23(4): 040403.

[5] R. Jun,J. Meng-Wen,W. Yi-Huan, *Invariant expression for tunneling rate under canonical transformations* Chinese Physics **B**, 2013, 22(11): 110401.

[6] L. Hui-Ling,L. Rong,C. Li-Ying. *Spectroscopy via adiabatic covariant action for the Banados-Teitelboim-Zanelli (BTZ) black hole*, Chinese Physics **B**, 2013, 22(5): 050402.

[7] L. Cheng-Zhou, *Area and entropy spectra of black holes via an adiabatic invariant*, Chinese Physics **B**, 2012, 21(7): 070401.

[8] L. Zhan-Fang,C. Gang,L. Ming-Jian, *Thermodynamic and geometric framework of a (2+1)-dimensional black hole with non-linear electrodynamics*, Chinese Physics **B**,2011,20(11):110404.

[9] Z. Jing-Yi, *Canonical Ensemble Model for the Black Hole Quantum Tunneling Radiation*, Chinese Physics Letters, 2013, **30**(7): 070401.

[10] J. Sadeghi and H. Farahani, *Thermodynamics of a charged hairy black hole in (2+1) dimensions*, Int. J. Theor. Phys. **53**, no. 11, 3683 (2014) [arXiv:1308.1054 [hep-th]].

[11] W. Xu, L. Zhao and D. C. Zou, *Three dimensional rotating hairy black holes, asymptotics and thermodynamics*, [arXiv:1406.7153 [gr-qc]].

[12] L. Zhao, W. Xu and B. Zhu, *Novel rotating hairy black hole in (2+1)-dimensions*, Commun. Theor. Phys. **61**, 475 (2014) [arXiv:1305.6001 [gr-qc]].

[13] W. Xu and L. Zhao, *Charged black hole with a scalar hair in (2+1) dimensions*, Phys. Rev. D **87**, no. 12, 124008 (2013) [arXiv:1305.5446 [gr-qc]].

[14] B. Pourhassan, *The Klein-Gordon Equation of a Rotating Charged Hairy Black Hole in (2+1) Dimensions*, arXiv:1506.08066 [hep-th].

[15] A. Belhaj, M. Chabab, H. E. Moumni, K. Masmar and M. B. Sedra, *Statistical Physics of 3D Hairy Black Holes*, [arXiv:1412.8322 [hep-th]].

[16] A. Belhaj, M. Chabab, H. El Moumni, K. Masmar, M. B. Sedra, *Critical Behaviors of 3D Black Holes with a Scalar Hair*, IJGMMP **12** 2 (2015) 1550017,[arXiv:1306.2518 [hep-th]].

[17] A. Belhaj, M. Chabab, H. El Moumni and M. B. Sedra, *On Thermodynamics of AdS Black Holes in Arbitrary Dimensions*, Chin. Phys. Lett. **29** (2012) 100401 [arXiv:1210.4617 [hep-th]].

[18] A. Belhaj, M. Chabab, H. E. Moumni, L. Medari and M. B. Sedra, *The Thermodynamical Behaviors of Kerr-Newman AdS Black Holes*, Chin. Phys. Lett. **30** (2013) 090402 [arXiv:1307.7421 [hep-th]].

[19] A. Belhaj, M. Chabab, H. El Moumni, K. Masmar, M. B. Sedra and A. Segui, *On Heat Properties of AdS Black Holes in Higher Dimensions*, JHEP **1505**, 149 (2015) [arXiv:1503.07308 [hep-th]].

[20] A. Belhaj, M. Chabab, H. El moumni, K. Masmar and M. B. Sedra, *Maxwell's equal-area law for Gauss-Bonnet-Anti-de Sitter black holes*, Eur. Phys. J. C **75**, no. 2, 71 (2015) [arXiv:1412.2162 [hep-th]].

[21] A. Belhaj, M. Chabab, H. EL Moumni, K. Masmar and M. B. Sedra, *Ehrenfest Scheme of Higher Dimensional Topological AdS Black Holes in The Third Order Lovelock-Born-Infeld Gravity*, [arXiv:1405.3306 [hep-th]].

[22] J. Liang and B. Liu, *Thermodynamics of non-commutative geometry inspired BTZ black hole based on Lorentzian smeared mass distribution*, EPL. **100** (2012) 30001.

[23] K. Nozari, A. Yazdani. The Energy Distribution of a Noncommutative Reissner-Nordstrom Black Hole. Chinese Physics Letters, 2013, **30** (9): 090401

[24] M. Hasanpour, F. Loran, and H. Razaghian, *Gravity/CFT correspondence for three dimensional Einstein gravity with a conformal scalar field*, Nucl. Phys. **867** (2013) 483.

[25] B. Chen, Z. Xue, and J. ju Zhang, *Note on Thermodynamic Method of Black Hole/CFT Correspondence*, JHEP **13**03 (2013) 102.

[26] S. Shankaranarayanan, K. Srinivasan and T. Padmanabhan , *Method of complex paths and general covariance of Hawking radiation*, Mod. Phys. Lett. A**16** 571 (2001), [arXiv: 0007022 [ge-qc]].

[27] K. Srinivasan and T. Padmanabhan, *Particle production and complex path analysis*, Phys. Rev. D **60** 024007 (1999), [arXiv: 9812028 [ge-qc]].

[28] A. Ejaz, H. Gohar, H. Lin, K. Saifullah and S. T. Yau, *Quantum tunneling from three-dimensional black holes*, Phys. Lett. B **726** (2013) 827 [arXiv:1306.6380 [hep-th]].